

# Duursma's reduced polynomial \*

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## Abstract

The weight distribution  $\{\mathcal{W}_C^{(w)}\}_{w=0}^n$  of a linear code  $C \subset \mathbb{F}_q^n$  is put in an explicit bijective correspondence with Duursma's reduced polynomial  $D_C(t) \in \mathbb{Q}[t]$  of  $C$ . We prove that the Riemann Hypothesis Analogue for a linear code  $C$  requires the formal self-duality of  $C$  and imposes an upper bound on the cardinality  $q$  of the basic field, depending on the dimension and the minimum distance of  $C$ . Duursma's reduced polynomial  $D_F(t) \in \mathbb{Z}[t]$  of the function field  $F = \mathbb{F}_q(X)$  of a curve  $X$  of genus  $g$  over  $\mathbb{F}_q$  is shown to provide a generating function  $\frac{D_F(t)}{(1-q)(1-qt)} = \sum_{i=0}^{\infty} \mathcal{B}_i t^i$  for the numbers  $\mathcal{B}_i$  of the effective divisors of degree  $i \geq 0$  of a virtual function field of a curve of genus  $g - 1$  over  $\mathbb{F}_q$ .

Let  $\overline{\mathbb{F}_q} = \cup_{m=1}^{\infty} \mathbb{F}_{q^m}$  be the algebraic closure of a finite field  $\mathbb{F}_q$  and  $X/\mathbb{F}_q \subset \mathbb{P}^N(\overline{\mathbb{F}_q})$  be a smooth irreducible projective curve of genus  $g$ , defined over  $\mathbb{F}_q$ . Denote by  $F = \mathbb{F}_q(X)$  the function field of  $X$  over  $\mathbb{F}_q$  and choose  $n$  different  $\mathbb{F}_q$ -rational points  $P_1, \dots, P_n \in X(\mathbb{F}_q) := X \cap \mathbb{P}^N(\mathbb{F}_q)$ . Suppose that  $G$  is an effective divisor of  $F$  of degree  $2g - 2 < \deg G = m < n$ , whose support is disjoint from the support of  $D = P_1 + \dots + P_n$ . The space  $L(G) := H^0(X, \mathcal{O}_X(G))$  of the global holomorphic sections of the line bundle, associated with  $G$  will be referred to as to the Riemann-Roch space of  $G$ . We put  $l(G) := \dim_{\mathbb{F}_q} L(G)$  and observe that the evaluation map

$$\mathcal{E}_D : L(G) \longrightarrow \mathbb{F}_q^n,$$

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$$\mathcal{E}_D(f) = (f(P_1), \dots, f(P_n)) \quad \text{for } \forall f \in L(G)$$

is an  $\mathbb{F}_q$ -linear embedding. Its image  $C := \text{im}(\mathcal{E}_D) = \mathcal{E}_D L(G)$  is known as an algebraic geometry code or Goppa code. The minimum distance of  $C$  is  $d(C) \geq n - m$ . For an arbitrary  $s \in \mathbb{N}$  let  $N_s(F) := |X(\mathbb{F}_{q^s})|$  be the number of the  $\mathbb{F}_{q^s}$ -rational points of  $X$ . Then the formal power series

$$Z_F(t) := \exp \left( \sum_{s=1}^{\infty} \frac{N_s(F)}{s} t^s \right)$$

is called the Hasse-Weil zeta function of  $F$ . It is well known (cf. Theorem 4.1.11 from [8]) that

$$Z_F(t) = \frac{L_F(t)}{(1-t)(1-qt)}$$

for a polynomial  $L_F(t) \in \mathbb{Z}[t]$  of degree  $2g$ . We refer to  $L_F(t)$  as to the Hasse-Weil polynomial of  $F$ .

In [2], [3] Duursma introduces the genus of a linear code  $C \subset \mathbb{F}_q^n$  as the deviation  $g := n + 1 - k - d$  of its dimension  $k := \dim_{\mathbb{F}_q} C$  and minimum distance  $d$  from the equality in Singleton bound. Let  $\mathcal{W}_C^{(w)}$  be the number of the codewords  $c \in C$  of weight  $d \leq w \leq n$ . Then

$$\mathcal{W}_C(x, y) := x^n + \sum_{w=d(C)}^n \mathcal{W}_C^{(w)} x^{n-w} y^w$$

is called the homogeneous weight enumerator of  $C$ . Denote by  $\mathcal{M}_{n,s}(x, y)$  the homogeneous weight enumerator of an MDS-code of length  $n$  and minimum distance  $s$ . Put  $g^\perp$  for the genus of the dual code  $C^\perp$  of  $C$  and  $r := g + g^\perp$ .

**Proposition 1.** (Duursma [3]) *For an arbitrary  $\mathbb{F}_q$ -linear  $[n, k, d]$ -code  $C$ , which is not contained in a coordinate hyperplane  $H_i := \{x \in \mathbb{F}_q^n \mid x_i = 0\}$  of  $\mathbb{F}_q^n$ , there exist uniquely determined rational numbers  $a_0, \dots, a_r \in \mathbb{Q}$ , such that the homogeneous weight enumerator*

$$\mathcal{W}_C(x, y) = a_0 \mathcal{M}_{n,d}(x, y) + a_1 \mathcal{M}_{n,d+1}(x, y) + \dots + a_r \mathcal{M}_{n,d+r}(x, y) \quad (1)$$

*of  $C$  is the linear combination of the homogeneous weight enumerators  $\mathcal{M}_{n,d+i}(x, y)$  of MDS-codes of length  $n$  and minimum distance  $d + i$  with coefficients  $a_i$  and*

$$P_C(1) = \sum_{i=0}^r a_i = 1. \quad (2)$$

*The  $\zeta$ -polynomial  $P_C(t) := \sum_{i=0}^r a_i t^i$  of  $C$  is uniquely determined by*

$$\frac{\mathcal{W}_C(x, y) - x^n}{q - 1} = \text{Coeff}_{t^{n-d}} \left( \frac{P_C(t)}{(1-t)(1-qt)} [y(1-t) + xt]^n \right), \quad (3)$$

*where  $\text{Coeff}_{t^{n-d}}(f(t))$  stands for the coefficient of  $t^{n-d}$  in a formal power series  $f(t) \in \mathbb{C}[[t]]$ .*

**Proposition 2.** (Duursma's considerations from [2]) Let  $X/\mathbb{F}_q \subset \mathbb{P}^N(\overline{\mathbb{F}_q})$  be a smooth irreducible curve of genus  $g$ , defined over  $\mathbb{F}_q$  and  $G_1, \dots, G_h$  be a complete list of effective representatives of the linear equivalence classes of the divisors of  $F = \mathbb{F}_q(X)$  of degree  $2g - 2 < m < n$ . Assume that there exist  $n$  different  $\mathbb{F}_q$ -rational points  $P_1, \dots, P_n \in X(\mathbb{F}_q)$ , such that  $D = P_1 + \dots + P_n \in \text{Div}(F)$  has support  $\text{Supp}(D) \cap \text{Supp}(G_i) = \{P_1, \dots, P_n\} \cap \text{Supp}(G_i) = \emptyset$  for  $\forall 1 \leq i \leq h$ . If

$$\mathcal{L}(G_i) = H^0(X, \mathcal{O}_X([G_i]) := \{f \in F^* \mid (f) + G_i \geq 0\} \cup \{0\}$$

are the Riemann-Roch spaces of  $G_i$ ,

$$\mathcal{E}_D : \mathcal{L}(G_i) \longrightarrow \mathbb{F}_q^n,$$

$$\mathcal{E}_D(f) = (f(P_1), \dots, f(P_n)) \quad \text{for } \forall f \in \mathcal{L}(G_i)$$

are the evaluation maps at  $D$  and  $C_i := \mathcal{E}_D \mathcal{L}(G_i)$  are the corresponding Goppa codes with homogeneous weight enumerators  $\mathcal{W}_{C_i}(x, y)$ , then

$$\sum_{i=1}^h \frac{\mathcal{W}_{C_i}(x, y) - x^n}{q - 1} = \text{Coeff}_{t^m} \left( \frac{L_F(t)}{(1-t)(1-qt)} [y(1-t) + xt]^n \right) \quad (4)$$

for the  $\zeta$ -polynomial  $L_F(t)$  of  $F$ .

In particular,

$$\sum_{i=1}^h t^{g-g_i} P_{C_i}(t) = L_F(t) \quad (5)$$

for the  $\zeta$ -polynomials  $P_{C_i}(t)$  of  $C_i = \mathcal{E}_D \mathcal{L}(G_i)$  and the Hasse-Weil polynomial  $L_F(T)$  of the function field  $F$ .

*Proof.* Note that (4) is an equality of homogeneous polynomials of  $x$  and  $y$  of degree  $n$ , whose monomials are of degree  $s \geq 1$  with respect to  $y$ . Therefore (4) is equivalent to

$$\begin{aligned} \frac{\sum_{i=1}^{h(F)} \mathcal{W}_{C_i}^{(s)}}{q-1} &= \text{Coeff}_{x^{n-s}y^s t^m} (\zeta_F(t) [y(1-t) + xt]^n) = \\ &= \text{Coeff}_{t^m} \left( \binom{n}{s} t^{n-s} (1-t)^s \zeta_F(t) \right) = \\ &= \binom{n}{s} \text{Coeff}_{t^{s-n+m}} ((1-t)^s \zeta_F(t)) \end{aligned} \quad (6)$$

for  $\forall s \in \mathbb{N}$ . Note that  $C_i$  are of minimum distance  $d(C_i) \geq n - m$ , so that  $\mathcal{W}_{C_i}^{(s)} = 0$  for  $1 \leq s < n - m$ . On the other hand,

$$(1-t)^s \zeta_F(t) = \frac{(1-t)^{s-1} L_F(t)}{1-qt}$$

has no pole at  $t = 0$ , so that  $\text{Coeff}_{t^{s-n+m}} ((1-t)^s \zeta_F(t)) = 0$  for  $s - n + m < 0$ ,  $s \in \mathbb{N}$ . That is why it suffices to verify (6) for  $s \geq n - m$ ,  $s \in \mathbb{N}$ .

Note that the number of the codewords  $c = (f(P_1), \dots, f(P_n)) \in C_i$ ,  $f \in L(G_i)$  of weight  $s$  equals the number of the rational functions  $f \in L(G_i) \setminus \{0\}$ , vanishing at  $n-s$  of the points  $P_1, \dots, P_n$ . Bearing in mind that the projective space  $\mathbb{P}(L(G_i)) = \mathbb{P}^{m-g}(\mathbb{F}_q)$  parameterizes the effective divisors, linearly equivalent to  $G_i$  and two rational functions  $f, f' \in F \setminus \{0\}$  have one and a same divisor exactly when they are on one and a same  $\mathbb{F}_q^*$ -orbit,  $f' \in \mathbb{F}_q^* f$ , one concludes that  $\frac{\mathcal{W}_{C_i}^{(s)}}{q-1}$  is the number of the effective divisors  $E = (f) + G_i$ , which are linearly equivalent to  $G_i$  with  $|\text{Supp}(E) \cap \text{Supp}(D)| = n-s$ . Thus,

$$e_{m,s} := \frac{\sum_{i=1}^{h(F)} \mathcal{W}_{C_i}^{(s)}}{q-1}$$

equals the number of the effective divisors  $E \in \text{Div}(F)^{\geq 0}$  of degree  $\deg E = m$  with  $|\text{Supp}(E) \cap \text{Supp}(D)| = n-s$ . For any  $s$ -tuple of indices  $i = \{i_1, \dots, i_s\}$ ,  $1 \leq i_1 < \dots < i_s \leq n$  let  $D_i := P_{i_1} + \dots + P_{i_s}$  and  $e_m(i)$  be the number of the effective divisors  $E \in \text{Div}(F)^{\geq 0}$  of degree  $\deg E = m$  with  $\text{Supp}(E) \cap \text{Supp}(D) = \text{Supp}(D - D_i)$ . Then  $e_{m,s} = \sum_i e_m(i)$  and it suffices to show that  $e_m(i) = \text{Coeff}_{t^{s-n+m}}((1-t)^s \zeta_F(t))$  for any  $i$ , in order to justify (6) and (4).

To this end, observe that  $E \in \text{Div}(F)^{\geq 0}$  is an effective divisor of degree  $\deg E = m$  with  $\text{Supp}(E) \cap \text{Supp}(D) = \text{Supp}(D - D_i)$  if and only if the difference  $E_1 := E - (D - D_i) \in \text{Div}(F)^{\geq 0}$  is an effective divisor of degree  $\deg E_1 = m - n + s$  with support  $\text{Supp}(E_1) \cap \text{Supp}(D_i) = \emptyset$ . Now,  $e_m(i)$  equals the number of the effective divisors  $E_1 \in \text{Div}(F)^{\geq 0}$  of degree  $\deg E_1 = m - n + s$  with  $\text{Supp}(E_1) \cap \text{Supp}(D_i) = \emptyset$ . Recall that the Hasse-Weil  $\zeta$ -function

$$\zeta_F(t) = \prod_{\nu \in \mathcal{P}} \frac{1}{1 - t^{\deg \nu}} = \sum_{i=0}^{\infty} \mathcal{A}_i t^i$$

is the generating function for the number  $\mathcal{A}_i$  of the effective divisors of  $F$  of degree  $i$ . Bearing in mind that  $D_i = \nu_{i_1} + \dots + \nu_{i_s}$  is a sum of  $s$  different places  $i_j$  of degree  $\deg \nu_{i_j} = 1$ , one observes that  $(1-t)^s \zeta_F(t)$  is the generating function for the number of the effective divisors of  $F$  of degree  $i$ , whose support is disjoint with  $\text{Supp}(D_i)$ . In other words,  $e_m(i) = \text{Coeff}_{t^{m-n+s}}((1-t)^s \zeta_F(t))$ .

The equality (5) is an immediate consequence of Proposition 1, (4) and the fact that  $C_i = \mathcal{E}_D \mathcal{L}(G_i)$  are of dimension  $\dim_{\mathbb{F}_q} C_i = l(G_i) = m - g + 1$ , minimum distance  $d_i \geq n - m$  and, therefore, of genus

$$g_i = n + 1 - \dim_{\mathbb{F}_q} C_i - d_i = n - m - d_i + g \leq g.$$

□

Proposition 2 motivates Duursma to refer to  $P_C(t)$  as to the zeta polynomial of an arbitrary linear code  $C \subset \mathbb{F}_q^n$ . He establishes that  $P_C(t)$  and  $\mathcal{W}_C(x, y)$  are in a bijective correspondence and Mac Williams identities, relating the weight distributions  $\{\mathcal{W}_C^{(w)}\}_{w=d}^n$ ,  $\{\mathcal{W}_{C^\perp}^{(w)}\}_{w=d^\perp}^n$  of a pair  $(C, C^\perp)$  of mutually dual linear codes are equivalent to the functional equation

$$P_{C^\perp}(t) = P_C\left(\frac{1}{qt}\right) q^g t^{g+g^\perp} \quad (7)$$

for the corresponding zeta polynomials  $P_C(t), P_{C^\perp}(t)$ .

In [2] and [4] Duursma observes the existence of a polynomial  $D_C(t) = \sum_{i=0}^{r-2} c_i t^i \in \mathbb{Q}[t]$ , defined by the identity

$$P_C(t) = (1-t)(1-qt)D_C(t) + t^g$$

of polynomials in  $t$ , but does not make use of  $D_C(t)$  for the study of the homogeneous weight enumerator  $\mathcal{W}_C(x, y)$  of  $C$ . He mentions in [4] that the analogue  $D_F(t)$  of  $D_C(t)$  for a function field  $F$  of one variable accounts for the contribution of the special divisors of  $F$  to the zeta function  $Z_F(t)$ . From now on, we refer to  $D_C(t)$  as to Duursma's reduced polynomial of  $C$ .

The present note provides an explicit bijective correspondence between the weight distribution  $\{\mathcal{W}_C^{(w)}\}_{w=d}^n$  of an arbitrary linear code  $C \subset \mathbb{F}_q^n$  and the coefficients  $\{c_i\}_{i=0}^{r-2}$  of its Duursma's reduced polynomial  $D_C(t) = \sum_{i=0}^{r-2} c_i t^i$  (cf. Proposition 3).

The classical Hasse-Weil Theorem establishes that all the roots of the Hasse-Weil polynomial  $L_F(t) \in \mathbb{Z}[t]$  of the function field  $F = \mathbb{F}_q(X)$  of a curve  $X$  of genus  $g$  over  $\mathbb{F}_q$  are on the circle  $S\left(\frac{1}{\sqrt{q}}\right) : \left\{z \in \mathbb{C} \mid |z| = \frac{1}{\sqrt{q}}\right\}$  (cf. Theorem 4.2.3 from [8]). Duursma says that a linear code  $C \subset \mathbb{F}_q^n$  satisfies the Riemann Hypothesis Analogue if all the roots of its zeta polynomial  $P_C(t) = \sum_{i=0}^r a_i t^i \in \mathbb{Q}[t]$  are on the circle  $S\left(\frac{1}{\sqrt{q}}\right)$ . Let  $C$  be an  $\mathbb{F}_q$ -linear code of dimension  $k$  and minimum distance  $d$ , which satisfies the Riemann Hypothesis Analogue. Proposition 4 shows that  $C$  is formally self-dual, while Corollary 5 provides an explicit upper bound on the cardinality  $q$  of the basic field, depending on  $k$  and  $d$ . Let us recall that  $C$  is formally self-dual if it has the same weight distribution  $\mathcal{W}_C^{(w)} = \mathcal{W}_{C^\perp}^{(w)}$ ,  $\forall 0 \leq w \leq n$  as its dual code  $C^\perp \subset \mathbb{F}_q^n$ . In the light of Duursma's results and our Proposition 3, the formal self-duality of  $C$  turns to be equivalent to the functional equation  $P_C(t) = P_C\left(\frac{1}{qt}\right) q^g t^{2g}$  for  $P_C(t)$  and to the functional equation  $D_C(t) = D_C\left(\frac{1}{qt}\right) q^{g-1} t^{2g-2}$  for  $D_C(t)$ . Proposition 6 from the present note expresses explicitly the homogeneous weight enumerator  $\mathcal{W}_C(x, y)$  of a formally self-dual code  $C \subset \mathbb{F}_q^n$  by the lowest half of the coefficients of  $D_C(t)$  or by the numbers  $\mathcal{W}_C^{(d)}, \dots, \mathcal{W}_C^{(k)}$  of the codewords  $c \in C$ , whose weights are between the minimum distance  $d$  of  $C$  and the dimension  $k$ .

In [1] Dodunekov and Landgev introduce the near-MDS code  $C \subset \mathbb{F}_q^n$  as the ones with quadratic zeta polynomial  $P_C(t)$ . Kim and Hyun's article [7] provides a necessary and sufficient condition for a near-MDS code to satisfy the Riemann Hypothesis Analogue. Note that the zeta polynomials  $P_C(t)$  and Duursma's reduced polynomials  $D_C(t)$  of formally self-dual codes  $C \subset \mathbb{F}_q^n$  are of even degree. Our Proposition 7 is a necessary and sufficient condition for a formally self-dual code  $C \subset \mathbb{F}_q^n$  with zeta polynomial  $P_C(t)$  of  $\deg P_C(t) = 4$  to be subject to the Riemann Hypothesis Analogue. Let  $S_\nu$ ,  $\nu \in \mathbb{N}$  be the uniquely determined logarithmic coefficients of  $P_C(t)$ , defined by the equality of formal power series  $\log P_C(t) = \sum_{\nu=1}^{\infty} S_\nu \frac{t^\nu}{\nu} \in \mathbb{C}[[t]]$ . Adapting Bombieri's proof of the

Hasse-Weil Theorem, [5] shows that a linear code  $\mathcal{C}$  satisfies the Riemann Hypothesis Analogue exactly when the sequence  $\{S_\nu q^{-\frac{\nu}{2}}\}_{\nu=1}^\infty \subset \mathbb{C}$  is absolutely bounded.

The last, third section is devoted to Duursma's reduced polynomial  $D_F(t)$  of the function field  $F = \mathbb{F}_q(X)$  of a curve  $X/\mathbb{F}_q \subset \mathbb{P}^N(\overline{\mathbb{F}}_q)$  of genus  $g$  over  $\mathbb{F}_q$ . It establishes that  $D_F(t) \in \mathbb{Z}[t]$  is determined uniquely by its lowest  $g$  coefficients, which equal the numbers  $\mathcal{A}_i$  of the effective divisors of  $F$  of degree  $0 \leq i \leq g-1$ . Our Proposition 9 shows that the zeta function

$$\frac{D_F(t)}{(1-t)(1-qt)} = \sum_{i=0}^{\infty} \mathcal{B}_i t^i,$$

associated with  $D_F(t)$  has the properties of a generating function for the numbers  $\mathcal{B}_i$  of the effective divisors of degree  $i \geq 0$  of a virtual function field of genus  $g-1$  over  $\mathbb{F}_q$ . There arises the following

**Open Problem:** To characterize the function fields  $F = \mathbb{F}_q(X)$  of curves  $X/\mathbb{F}_q \subset \mathbb{P}^N(\overline{\mathbb{F}}_q)$  of genus  $g$  over  $\mathbb{F}_q$ , for which there are curves  $Y/\mathbb{F}_q \subset \mathbb{P}^M(\overline{\mathbb{F}}_q)$  of genus  $g-1$ , defined over  $\mathbb{F}_q$  with Hasse-Weil zeta function

$$Z_{\mathbb{F}_q(Y)}(t) = \frac{D_F(t)}{(1-t)(1-qt)}.$$

## 1 The homogeneous weight enumerator of an arbitrary code

**Proposition 3.** Let  $C \subset \mathbb{F}_q^n$  be a linear code of dimension  $k = \dim_{\mathbb{F}_q} C$ , minimum distance  $d$  and genus  $g = n+1-k-d \geq 1$ , whose dual  $C^\perp \subset \mathbb{F}_q^n$  is of minimum distance  $d^\perp$  and genus  $g^\perp = k+1-d^\perp \geq 1$ . If

$$D_C(t) = \sum_{i=0}^{g+g^\perp-2} c_i t^i \in \mathbb{Q}[t]$$

is Duursma's reduced polynomial of  $C$  and  $\mathcal{M}_{n,n+1-k}(x,y)$  is the homogeneous weight enumerator of an MDS-code of length  $n$ , dimension  $k$  and minimum distance  $n+1-k$ , then the homogeneous weight enumerator of  $C$  is

$$\mathcal{W}_C(x,y) = \mathcal{M}_{n,n+1-k}(x,y) + (q-1) \sum_{i=0}^{g+g^\perp-2} c_i \binom{n}{d+i} (x-y)^{n-d-i} y^{d+i}. \quad (8)$$

More precisely, Duursma's reduced polynomial  $D_C(t) = \sum_{i=0}^{g+g^\perp-2} c_i t^i$  determines uniquely the weight distribution of  $C$ , according to

$$\mathcal{W}_C^{(w)} = (q-1) \binom{n}{w} \sum_{i=0}^{w-d} (-1)^{w-d-i} \binom{w}{d+i} c_i \quad \text{for } d \leq w \leq d+g-1, \quad (9)$$

$$\begin{aligned} \mathcal{W}_C^{(w)} = & (q-1) \binom{n}{w} \sum_{i=0}^{\min(w-d, n-d-d^\perp)} (-1)^{w-d-i} \binom{w}{d+i} c_i \\ & + \binom{n}{w} \sum_{j=0}^{w-n-1+k} (-1)^j \binom{w}{j} (q^{w-n+k-j} - 1) \quad \text{for } d+g \leq w \leq n. \end{aligned} \quad (10)$$

Conversely, for  $\forall 0 \leq i \leq g+g^\perp-2$  the numbers  $\mathcal{W}_C^{(d)}, \dots, \mathcal{W}_C^{(d+i)}$  determine uniquely the coefficient  $c_i$  of Duursma's reduced polynomial  $D_C(t) = \sum_{i=0}^{g+g^\perp-2} c_i t^i$  by

$$c_i = (q-1)^{-1} \binom{n}{d+i}^{-1} \sum_{w=d}^{d+i} \binom{n-w}{n-d-i} \mathcal{W}_C^{(w)} \quad (11)$$

for  $0 \leq i \leq g-1$ ,

$$\begin{aligned} c_i = & (q-1)^{-1} \binom{n}{d+i}^{-1} \left\{ \sum_{w=d}^{d+g-1} \binom{n-w}{n-d-i} \mathcal{W}_C^{(w)} \right. \\ & \left. + \sum_{w=d+g}^{d+i} \binom{n-w}{n-d-i} \left[ \mathcal{W}_C^{(w)} - \binom{n}{w} \sum_{j=0}^{w-n-1+k} (-1)^j \binom{w}{j} (q^{w-n+k-j} - 1) \right] \right\} \end{aligned} \quad (12)$$

for  $g \leq i \leq g+g^\perp-2$ .

In particular,

$$(q-1) \binom{n}{d+i} c_i \in \mathbb{Z}$$

are integers for all  $0 \leq i \leq g+g^\perp-2$ .

The aforementioned formulae imply that  $\mathcal{W}_C^{(d)}, \dots, \mathcal{W}_C^{(d+g+g^\perp-2)}$  determine uniquely the homogeneous weight enumerator  $\mathcal{W}_C(x, y)$  of  $C$  by the formula

$$\mathcal{W}_C(x, y) = \sum_{w=d}^{d+g+g^\perp-2} \mathcal{W}_C^{(w)} \lambda_w(x, y) + \Lambda(x, y), \quad (13)$$

with explicit polynomials

$$\lambda_w(x, y) := \sum_{s=w}^{d+g+g^\perp-2} \binom{n-w}{n-s} (x-y)^{n-s} y^s \quad \text{for } d \leq w \leq d+g+g^\perp-2 \quad (14)$$

and

$$\Lambda(x, y) := \mathcal{M}_{n, n+1-k}(x, y) - \sum_{w=d+g}^{d+g+g^\perp-2} \mathcal{M}_{n, n+1-k}^{(w)} \lambda_w(x, y). \quad (15)$$

*Proof.* In the case of  $g = 0$ , note that  $C$  is an MDS-code and  $\mathcal{W}_C(x, y) = \mathcal{M}_{n, n+1-k}(x, y)$ . From now on, we assume that  $g > 0$  and put  $r := g + g^\perp$ . Making use of  $d + g = n + 1 - k$ , let us express

$$\mathcal{W}_C(x, y) = \mathcal{M}_{n, d+g}(x, y) + \sum_{i=0}^r b_i \mathcal{M}_{n, d+i}(x, y)$$

by some rational numbers  $b_i \in \mathbb{Q}$ . Then the seta polynomial  $P_C(t) = t^g + \sum_{i=0}^r b_i t^i$  and

Duursma's reduced polynomial  $D_C(t) = \sum_{i=0}^{r-2} c_i t^i$  of  $C$  are related by the equality

$$P_C(t) - t^g = (1 - t)(1 - qt)D_C(t). \quad (16)$$

Let us introduce  $c_{-2} = c_{-1} = c_{r-1} = c_r = 0$  and compare the coefficients of  $t^i$  from the left and right hand side of (16), in order to obtain

$$b_i = c_i - (q + 1)c_{i-1} + qc_{i-2} \quad \text{for } \forall 0 \leq i \leq r.$$

Therefore

$$\begin{aligned} \mathcal{W}_C(x, y) &= \mathcal{M}_{n, d+g}(x, y) + \sum_{i=0}^r c_i \mathcal{M}_{n, d+i}(x, y) \\ &\quad - (q + 1) \sum_{i=0}^r c_{i-1} \mathcal{M}_{n, d+i}(x, y) + q \sum_{i=0}^r c_{i-2} \mathcal{M}_{n, d+i}(x, y). \end{aligned}$$

Setting  $j = i - 1$ , respectively,  $j = i - 2$  in the last two sums, one obtains

$$\begin{aligned} \mathcal{W}_C(x, y) &= \mathcal{M}_{n, d+g}(x, y) + \sum_{i=0}^r c_i \mathcal{M}_{n, d+i}(x, y) \\ &\quad - (q + 1) \sum_{j=-1}^{r-1} c_j \mathcal{M}_{n, d+j+1}(x, y) + q \sum_{j=-2}^{r-2} c_j \mathcal{M}_{n, d+j+2}(x, y), \end{aligned}$$

whereas

$$\begin{aligned} \mathcal{W}_C(x, y) &= \mathcal{M}_{n, d+g}(x, y) \\ &\quad + \sum_{j=0}^{r-2} c_j [\mathcal{M}_{n, d+j}(x, y) - (q + 1)\mathcal{M}_{n, d+j+1}(x, y) + q\mathcal{M}_{n, d+j+2}(x, y)]. \end{aligned} \quad (17)$$

Let us put

$$\mathcal{W}_{n, d+j}(x, y) := \mathcal{M}_{n, d+j}(x, y) - (q + 1)\mathcal{M}_{n, d+j+1}(x, y) + q\mathcal{M}_{n, d+j+2}(x, y)$$

and recall that the homogeneous weight enumerator of an MDS-code of length  $n$  and minimum distance  $d + j$  is

$$\mathcal{M}_{n, d+j}(x, y) = x^n + \sum_{w=d+j}^n \mathcal{M}_{n, d+j}^{(w)} x^{n-w} y^w$$



with

$$\mathcal{M}_{n,d+j}^{(w)} = \binom{n}{w} \sum_{i=0}^{w-d-j} (-1)^i \binom{w}{i} (q^{w+1-d-j-i} - 1). \quad (18)$$

Therefore

$$\begin{aligned} \mathcal{W}_{n,d+j}(x, y) &= \mathcal{M}_{n,d+j}^{(d+j)} x^{n-d-j} y^{d+j} + [\mathcal{M}_{n,d+j}^{(d+j+1)} - (q+1)\mathcal{M}_{n,d+j+1}^{(d+j+1)}] x^{n-d-j-1} y^{d+j+1} \\ &\quad + \sum_{w=d+j+2}^n [\mathcal{M}_{n,d+j}^{(w)} - (q+1)\mathcal{M}_{n,d+j+1}^{(w)} + q\mathcal{M}_{n,d+j+2}^{(w)}] x^{n-w} y^w. \end{aligned}$$

Making use of the weight distribution (18) of an MDS-code and introducing

$$\mathcal{W}_{n,d+j}^{(w)} := \mathcal{M}_{n,d+j}^{(w)} - (q+1)\mathcal{M}_{n,d+j+1}^{(w)} + q\mathcal{M}_{n,d+j+2}^{(w)} \quad \text{for } d+j+2 \leq w \leq n,$$

one expresses

$$\begin{aligned} \mathcal{W}_{n,d+j}(x, y) &= \binom{n}{d+j} (q-1) x^{n-d-j} y^{d+j} \\ &\quad - \binom{n}{d+j+1} (q-1)(d+j+1) x^{n-d-j-1} y^{d+j+1} + \sum_{w=d+j+2}^n \mathcal{W}_{n,d+j}^{(w)} x^{n-w} y^w. \end{aligned}$$

For any  $d+j+2 \leq w \leq n$  one has

$$\mathcal{W}_{n,d+j}^{(w)} = \binom{n}{w} \binom{w}{d+j} (q-1) (-1)^{w-d-j}.$$

Making use of

$$\binom{n}{w} \binom{w}{d+j} = \binom{n-d-j}{w-d-j} \binom{n}{d+j},$$

one obtains

$$\begin{aligned} \mathcal{W}_{n,d+j}(x, y) &= \binom{n}{d+j} (q-1) x^{n-d-j} y^{d+j} - \binom{n}{d+j+1} (q-1)(d+j+1) x^{n-d-j-1} y^{d+j+1} + \\ &\quad + \sum_{w=d+j+2}^n \binom{n}{d+j} \binom{n-d-j}{w-d-j} (q-1) (-1)^{w-d-j} x^{n-w} y^w. \end{aligned}$$

Bearing in mind that

$$(d+j+1) \binom{n}{d+j+1} = (n-d-j) \binom{n}{d+j},$$

one derives that

$$\begin{aligned} \mathcal{W}_{n,d+j}(x, y) &= \binom{n}{d+j} (q-1) \left[ x^{n-d-j} y^{d+j} - (n-d-j) x^{n-d-j-1} y^{d+j+1} + \right. \\ &\quad \left. + \sum_{w=d+j+2}^n (-1)^{w-d-j} \binom{n-d-j}{w-d-j} x^{n-w} y^w \right]. \end{aligned}$$

Introducing  $s := w - d - j$ , one expresses

$$\sum_{w=d+j+2}^n (-1)^{w-d-j} \binom{n-d-j}{w-d-j} x^{n-w} y^w = \sum_{s=2}^{n-d-j} (-1)^s \binom{n-d-j}{s} x^{n-d-j-s} y^{d+j+s}$$

and concludes that

$$\mathcal{W}_{n,d+j}(x, y) = \binom{n}{d+j} (q-1) (x-y)^{n-d-j} y^{d+j}. \quad (19)$$

The equality  $\mathcal{W}_{n,n-k}(x, y) = \binom{n}{k} (q-1) (x-y)^k y^{n-k}$  is exactly the claim (c) of Lemma 1 from Kim and Nyun's work [7]. Plugging in (19) in (17) and bearing in mind that  $d+g = n+1-k$ , one obtains (8).

In order to prove (9) and (10), let us put

$$\mathcal{V}_C(x, y) := \mathcal{W}_C(x, y) - \mathcal{M}_{n,n+1-k}(x, y)$$

and note that  $\mathcal{V}_C(x, y) = \sum_{w=d}^n \mathcal{V}_C^{(w)} x^{n-w} y^w$  with  $\mathcal{V}_C^{(w)} = \mathcal{W}_C^{(w)}$  for  $d \leq w \leq n-k$ ,

$$\mathcal{V}_C^{(w)} = \mathcal{W}_C^{(w)} - \mathcal{M}_{n,n+1-k}^{(w)} = \mathcal{W}_C^{(w)} - \binom{n}{w} \sum_{i=0}^{w-n+1+k} (-1)^i \binom{w}{i} (q^{w-n+k-i} - 1)$$

for  $d+g = n+1-k \leq w \leq n$ . Making use of (8), one expresses

$$\begin{aligned} \mathcal{V}_C(x, y) &= (q-1) \sum_{i=0}^{g+g^\perp-2} c_i \binom{n}{d+i} \sum_{s=0}^{n-d-i} \binom{n-d-i}{s} (-1)^{n-d-i-s} x^s y^{n-s} \\ &= (q-1) \sum_{s=0}^{n-d} \left[ \sum_{i=0}^{\min(n-d-s, g+g^\perp-2)} c_i \binom{n}{d+i} \binom{n-d-i}{s} (-1)^{n-d-i-s} \right] x^s y^{n-s}, \end{aligned}$$

after changing the summation order. Setting  $w := n-s$ , one obtains

$$\mathcal{V}_C(x, y) = (q-1) \sum_{w=d}^n \left[ \sum_{i=0}^{\min(w-d, n-d-d^\perp)} c_i \binom{n}{d+i} \binom{n-d-i}{n-w} (-1)^{w-d-i} \right] x^{n-w} y^w.$$

Then

$$\binom{n}{d+i} \binom{n-d-i}{n-w} = \binom{n}{w} \binom{w}{d+i},$$

allows to concludes that

$$\mathcal{V}_C^{(w)} = (q-1) \binom{n}{w} \sum_{i=0}^{\min(w-d, n-d-d^\perp)} c_i \binom{w}{d+i} (-1)^{w-d-i} \quad \text{for } \forall d \leq w \leq n,$$

which proves (9), (10).

Towards (11), (12), let us introduce  $z := x - y$  and express (8) in the form

$$\mathcal{V}_C(y + z, y) = (q - 1) \sum_{i=0}^{g+g^\perp-2} c_i \binom{n}{d+i} z^{n-d-i} y^{d+i}. \quad (20)$$

On the other hand,

$$\begin{aligned} \mathcal{V}_C(y + z, y) &= \sum_{w=d}^n \mathcal{V}_C^{(w)}(y + z)^{n-w} y^w \\ &= \sum_{w=d}^n \sum_{s=0}^{n-w} \binom{n-w}{s} \mathcal{V}_C^{(w)} y^{n-s} z^s = \sum_{s=0}^{n-d} \left[ \sum_{w=d}^{n-s} \binom{n-w}{s} \mathcal{V}_C^{(w)} \right] y^{n-s} z^s, \end{aligned}$$

after changing the summation order. Comparing the coefficients of  $y^{d+i} z^{n-d-i}$  in the left and right hand side of (20), one obtains

$$\sum_{w=d}^{d+i} \binom{n-w}{n-d-i} \mathcal{V}_C^{(w)} = (q-1) c_i \binom{n}{d+i},$$

whereas

$$c_i = (q-1)^{-1} \binom{n}{d+i}^{-1} \sum_{w=d}^{d+i} \binom{n-w}{n-d-i} \mathcal{V}_C^{(w)}.$$

Combining with (18), one justifies (11) and (12). These formulae imply also the fact that  $(q-1) \binom{n}{d+i} c_i \in \mathbb{Z}$  are integers for all  $0 \leq i \leq g + g^\perp - 2$ .

The substitution by (11), (12), (18) in (8) yields

$$\begin{aligned} \mathcal{W}_C(x, y) &= \mathcal{M}_{n, n+1-k}(x, y) + \sum_{i=0}^{g+g^\perp-2} \sum_{w=d}^{d+i} \binom{n-w}{n-d-i} \mathcal{W}_C^{(w)}(x-y)^{n-d-i} y^{d+i} \\ &\quad - \sum_{i=g}^{g+g^\perp-2} \sum_{w=d+g}^{d+i} \binom{n-w}{n-d-i} \mathcal{M}_{n, n+1-k}^{(w)}(x-y)^{n-d-i} y^{d+i}. \end{aligned}$$

One exchanges the summation order in the double sums towards

$$\begin{aligned} \mathcal{W}_C(x, y) &= \mathcal{M}_{n, n+1-k}(x, y) + \sum_{w=d}^{d+g+g^\perp-2} \mathcal{W}_C^{(w)} \sum_{i=w-d}^{g+g^\perp-2} \binom{n-w}{n-d-i} (x-y)^{n-d-i} y^{d+i} \\ &\quad - \sum_{w=d+g}^{d+g+g^\perp-2} \mathcal{M}_{n, n+1-k}^{(w)} \sum_{i=w-d}^{g+g^\perp-2} \binom{n-w}{n-d-i} (x-y)^{n-d-i} y^{d+i}. \end{aligned}$$

Introducing  $s := d + i$ , one obtains (13) with (14) and (15). □

Comparing the coefficients of  $x^{n-d}y^d$  in the left and right hand sides of (8), one obtains  $\mathcal{W}_C^{(d)} = (q-1)\binom{n}{d}c_0$  for a linear code  $C$  of genus  $g \geq 1$ . We claim that  $c_0 < 1$ . To this end, note that for any  $d$ -tuple  $\{i_1, \dots, i_d\} \subset \{1, \dots, n\}$ , supporting a word  $c \in C$  of weight  $d$  there are exactly  $q-1$  words  $c' \in C$  with  $\text{Supp}(c') = \text{Supp}(c) = \{i_1, \dots, i_d\}$ . That is due to the fact that the columns  $H_{i_1}, \dots, H_{i_d}$  of an arbitrary parity check matrix  $H$  of  $C$  are of rank  $d-1$  and there are no words of weight  $\leq d-1$  in the right null space of the matrix  $(H_{i_1} \dots H_{i_d})$ . It is clear that  $\nu \leq \binom{n}{d}$ , so that

$$c_0 = \frac{\nu}{\binom{n}{d}} \leq 1.$$

If we assume that  $c_0 = 1$  then any  $d$ -tuple of columns of  $H$  is linearly dependent. Bearing in mind that  $\text{rk} H = n - k$ , one concludes that  $d > n - k$ . Combining with Singleton Bound  $d \leq n - k + 1$ , one obtains  $d = n - k + 1$ . That contradicts the assumption that  $C$  is not an MDS-code and proves that  $c_0 < 1$  for any  $\mathbb{F}_q$ -linear code  $C \subset \mathbb{F}_q^n$  of genus  $g \geq 1$ . Note that  $c_0$  can be interpreted as the probability for a  $d$ -tuple to support a word of weight  $d$  from  $C$ .

## 2 The Riemann Hypothesis Analogue and the formal self-duality of a linear code

Recall that a linear code  $C \subset \mathbb{F}_q^n$  with dual code  $C^\perp \subset \mathbb{F}_q^n$  is formally self-dual if  $C$  and  $C^\perp$  have one and a same number  $\mathcal{W}_C^{(w)} = \mathcal{W}_{C^\perp}^{(w)}$  of codewords of weight  $0 \leq w \leq n$ . Let us mention some trivial consequences of the formal self-duality of  $C$ . First of all,  $C$  and  $C^\perp$  have one and a same minimum distance  $d = d(C) = d(C^\perp) = d^\perp$ . Further,  $C$  and  $C^\perp$  have one and a same cardinality

$$q^{\dim C} = \sum_{w=0}^n \mathcal{W}_C^{(w)} = \sum_{w=0}^n \mathcal{W}_{C^\perp}^{(w)} = q^{\dim C^\perp},$$

so that  $k = \dim C = \dim C^\perp = k^\perp$  and the length  $n = k + k^\perp = 2k$  is an even integer.

The genera  $g = k + 1 - d = g^\perp$  also coincide. Let  $P_C(t) = \sum_{i=0}^{2g} a_i t^i$  and  $P_{C^\perp} = \sum_{i=0}^{2g} a_i^\perp t^i$

be the zeta polynomials of  $C$ , respectively, of  $C^\perp$ . The consecutive comparison of the coefficients of  $x^{n-d}y^d, x^{n-d-1}y^{d+1}, \dots, x^{n-d-2g}y^{d+2g}$  from the homogeneous polynomial

$$\begin{aligned} & a_0 \mathcal{M}_{2k,d}(x, y) + a_1 \mathcal{M}_{2k,d+1}(x, y) + \dots + a_{2g} \mathcal{M}_{2k,d+2g}(x, y) = \mathcal{W}_C(x, y) \\ & = \mathcal{W}_{C^\perp}(x, y) = a_0^\perp \mathcal{M}_{2k,d}(x, y) + a_1^\perp \mathcal{M}_{2k,d+1}(x, y) + \dots + a_{2g}^\perp \mathcal{M}_{2k,d+2g}(x, y) \end{aligned}$$

in  $x, y$  yields  $a_i = a_i^\perp$  for  $\forall 0 \leq i \leq 2g$ . It is clear that  $a_i = a_i^\perp$  for  $\forall 0 \leq i \leq 2g$  suffices for  $\mathcal{W}_C(x, y) = \mathcal{W}_{C^\perp}(x, y)$ , so that the formal self-duality of  $C$  is tantamount to the coincidence  $P_C(t) = P_{C^\perp}(t)$  of the zeta polynomials of  $C$  and  $C^\perp$ . Duursma has shown that Mac Williams identities for  $\mathcal{W}_C^{(w)}$  and  $\mathcal{W}_{C^\perp}^{(w)}$  are equivalent to the functional equation (7) for the zeta polynomials  $P_C(t), P_{C^\perp}(t)$  of  $C, C^\perp \subset \mathbb{F}_q^n$  with genera  $g, g^\perp$ .

Thus, an  $\mathbb{F}_q$ -linear code  $C \subset \mathbb{F}_q^n$  is formally self-dual if and only if its zeta polynomial  $P_C(t)$  satisfies the functional equation

$$P_C(t) = P_C\left(\frac{1}{qt}\right) q^g t^{2g} \quad (21)$$

of the Hasse-Weil polynomial of the function field of a curve of genus  $g$  over  $\mathbb{F}_q$ .

**Proposition 4.** *If a linear code  $C \subset \mathbb{F}_q^n$  satisfies the Riemann Hypothesis Analogue then  $C$  is formally self-dual, i.e., the zeta polynomial  $P_C(t)$  of  $C$  is subject to the functional equation (21) of the Hasse-Weil polynomial of the function field of a curve of genus  $g$  over  $\mathbb{F}_q$ .*

*Proof.* Let us assume that  $P_C(t)$  of degree  $r := g + g^\perp$  satisfies the Riemann Hypothesis Analogue, i.e.,

$$P_C(t) = a_r \prod_{j=1}^r (t - \alpha_j) \in \mathbb{Q}[t]$$

for some  $\alpha_j \in \mathbb{C}$  with  $|\alpha_j| = \frac{1}{\sqrt{q}}$  for all  $1 \leq j \leq r$ . If  $\alpha_j$  is a real root of  $P_C(t)$  then  $\alpha_j = \frac{\varepsilon}{\sqrt{q}}$  with  $\varepsilon = \pm 1$ . We claim that in the case of an even degree  $r = 2m$ , the zeta polynomial  $P_C(t)$  is of the form

$$P_C(t) = a_{2m} \prod_{i=1}^m (t - \alpha_i)(t - \overline{\alpha_i}) \quad (22)$$

or of the form

$$P_C(t) = a_{2m} \left(t^2 - \frac{1}{q}\right) \prod_{i=1}^{m-1} (t - \alpha_i)(t - \overline{\alpha_i}), \quad (23)$$

while for an odd degree  $r = 2m + 1$  one has

$$P_C(t) = a_{2m+1} \left(t - \frac{\varepsilon}{\sqrt{q}}\right) \prod_{i=1}^m (t - \alpha_i)(t - \overline{\alpha_i}) \quad (24)$$

for some  $\varepsilon \in \{\pm 1\}$ . Indeed, if  $\alpha_i \in \mathbb{C} \setminus \mathbb{R}$  is a complex, non-real root of  $P_C(t) \in \mathbb{Q}[t] \subset \mathbb{R}[t]$  then  $\overline{\alpha_i} \neq \alpha_i$  is also a root of  $P_C(t)$  and  $P_C(t)$  is divisible by  $(t - \alpha_i)(t - \overline{\alpha_i})$ . If  $P_C(t) = 0$  has three real roots  $\alpha_1, \alpha_2, \alpha_3 \in \left\{\frac{1}{\sqrt{q}}, -\frac{1}{\sqrt{q}}\right\}$ , then at least two of them coincide. For  $\alpha_1 = \alpha_2 = \frac{\varepsilon}{\sqrt{q}}$  one has  $(t - \alpha_1)(t - \alpha_2) = (t - \alpha_1)(t - \overline{\alpha_1})$ . Thus,  $P_C(t)$  has at most two real roots, which are not complex conjugate (or, equivalently, equal) to each other and  $P_C(t)$  is of the form (22), (23) or (24).

If  $P_C(t)$  is of the form (22), then  $P_C(t) = a_{2m} \prod_{i=1}^m \left(t^2 - 2\operatorname{Re}(\alpha_i)t + \frac{1}{q}\right)$  and (7) reads as

$$P_{C^\perp}(t) = a_{2m} \left[ \prod_{i=1}^m \left(\frac{1}{q} - 2\operatorname{Re}(\alpha_i)t + t^2\right) \right] q^{g-m} = P_C(t) q^{g-m}, \quad (25)$$

after multiplying each of the factors  $\frac{1}{q^2 t^2} - \frac{2\operatorname{Re}(\alpha_i)}{qt} + \frac{1}{q}$  by  $qt^2$ . If  $D_C(t)$  is Duursma's reduced polynomial of  $C$  and  $D_{C^\perp}(t)$  is Duursma's reduced polynomial of  $C^\perp$ , then

$$(1-t)(1-qt)D_{C^\perp}(t) + t^{g^\perp} = P_{C^\perp}(t) = P_C(t)q^{g-m} = (1-t)(1-qt)q^{g-m}D_C(t) + q^{g-m}t^g$$

implies that

$$(1-t)(1-qt)[D_{C^\perp}(t) - q^{g-m}D_C(t)] = q^{g-m}t^g - t^{g^\perp}.$$

Plugging in  $t = 1$ , one concludes that  $q^{g-m} = 1$ , whereas  $g = m$ . As a result,  $g + g^\perp = 2m = 2g$  specifies that  $g = g^\perp$  and (25) yields  $P_C(t) = P_{C^\perp}(t)$ , which is equivalent to the formal self-duality of  $C$ .

If  $P_C(t)$  is of the form (23) then (7) provides

$$P_{C^\perp}(t) = a_{2m} \left( \frac{1}{q} - t^2 \right) \left[ \prod_{i=1}^{m-1} \left( \frac{1}{q} - 2\operatorname{Re}(\alpha_i)t + t^2 \right) \right] q^{g-m} = -P_C(t)q^{g-m}. \quad (26)$$

Expressing by Duursma's reduced polynomials  $D_C(t), D_{C^\perp}(t)$ , one obtains

$$(1-t)(1-qt)D_{C^\perp}(t) + t^{g^\perp} = P_{C^\perp}(t) = -P_C(t)q^{g-m} = -(1-t)(1-qt)q^{g-m}D_C(t) - q^{g-m}t^g,$$

whereas

$$(1-t)(1-qt)[D_{C^\perp}(t) + q^{g-m}D_C(t)] = -t^{g^\perp} - q^{g-m}t^g.$$

The substitution  $t = 1$  in the last equality of polynomials yields  $-1 - q^{g-m} = 0$ , which is an absurd, justifying that a zeta polynomial  $P_C(t)$ , subject to the Riemann Hypothesis Analogue cannot be of the form (23).

If  $P_C(t)$  is of odd degree  $2m + 1$ , then (24) and (7) yield

$$\begin{aligned} P_{C^\perp}(t) &= -\varepsilon\sqrt{q}a_{2m+1} \left( t - \frac{\varepsilon}{\sqrt{q}} \right) \left[ \prod_{i=1}^m \left( \frac{1}{q} - 2\operatorname{Re}(\alpha_i)t + t^2 \right) \right] q^{g-m-1} \\ &= -\varepsilon\sqrt{q}P_C(t)q^{g-m-1} \end{aligned}$$

after multiplying  $\frac{1}{qt} - \frac{\varepsilon}{\sqrt{q}}$  by  $-\frac{\varepsilon}{\sqrt{q}}qt$  and each  $\frac{1}{q^2 t^2} - \frac{2\operatorname{Re}(\alpha_i)}{qt} + \frac{1}{q}$  by  $qt^2$ . Expressing by Duursma's reduced polynomials

$$\begin{aligned} (1-t)(1-qt)D_{C^\perp}(t) + t^{g^\perp} &= P_{C^\perp}(t) = -\varepsilon q^{g-m-\frac{1}{2}}P_C(t) \\ &= -\varepsilon q^{g-m-\frac{1}{2}}(1-t)(1-qt)D_C(t) - \varepsilon q^{g-m-\frac{1}{2}}t^g, \end{aligned}$$

one obtains

$$(1-t)(1-qt) \left[ D_{C^\perp}(t) + \varepsilon q^{g-m-\frac{1}{2}}D_C(t) \right] = -t^{g^\perp} - \varepsilon q^{g-m-\frac{1}{2}}t^g.$$

The substitution  $t = 1$  implies  $-1 - \varepsilon q^{g-m-\frac{1}{2}} = 0$ , which is an absurd, as far as  $q^x = 1$  if and only if  $x = 0$ , while  $g - m - \frac{1}{2}$  cannot vanish for integers  $g, m$ . Thus, none zeta polynomial of odd degree satisfies the Riemann Hypothesis Analogue.  $\square$

**Corollary 5.** *If an  $\mathbb{F}_q$ -linear code  $C$  of  $\dim_{\mathbb{F}_q} C = k$  and minimum distance  $d$  satisfies the Riemann Hypothesis Analogue then the cardinality  $q$  of the basic field satisfies the upper bound*

$$q \leq \left( \sqrt[2g]{\binom{2k}{d}} + 1 \right)^2.$$

*Proof.* By Proposition 4, if  $C$  satisfies the Riemann Hypothesis Analogue then

$$P_C(t) = a_{2g} \prod_{j=1}^q \left( t - \frac{e^{i\varphi_j}}{\sqrt{q}} \right) \left( t - \frac{e^{-i\varphi_j}}{\sqrt{q}} \right)$$

for some  $\varphi_j \in [0, 2\pi)$ . The formal self-duality of  $C$  is equivalent to the functional equation  $P_C(t) = P_C\left(\frac{1}{qt}\right) q^g t^{2g}$  of the Hasse-Weil polynomial of a function field of genus  $g$  over  $\mathbb{F}_q$  and implies that  $a_{2g} = q^g a_0$ . Comparing the coefficients of  $x^{2k-d} y^d$  in the expression

$$\mathcal{W}_C(x, y) = a_0 \mathcal{M}_{2k,d}(x, y) + a_1 \mathcal{M}_{2k,d+1}(x, y) + \dots + a_{2g} \mathcal{M}_{2k,d+2g}(x, y)$$

of the homogeneous weight enumerator  $\mathcal{W}_C(x, y)$  of  $C$  by the homogeneous weight enumerators  $\mathcal{M}_{2k,d+i}(x, y)$  of MDS-codes of length  $2k$  and minimum distance  $d + i$ , one concludes that  $\mathcal{W}_C^{(d)} = a_0 \mathcal{M}_{2k,d}^{(d)} = a_0 (q-1) \binom{2k}{d}$ . Note that any word  $c \in C$  of weight  $d$  is a solution of a homogeneous linear system of rank  $d-1$  in  $d$  variables, as far as any  $d-1$  columns of a parity check matrix of  $C$  are linearly independent. Thus, there are exactly  $q-1$  words of weight  $d$  from  $C$  with the same support as  $c$ . If  $\nu$  is the number of the  $d$ -tuples, supporting a word  $c \in C$  of weight  $d$  then  $\mathcal{W}_C^{(d)} = (q-1)\nu$  and

$$a_0 = \frac{\nu}{\binom{2k}{d}}$$

is the probability for a  $d$ -tuple to support a word of weight  $d$  from  $C$ . Altogether, one obtains that

$$\begin{aligned} P_C(t) &= \frac{q^g \nu}{\binom{2k}{d}} \prod_{j=1}^q \left( t - \frac{e^{i\varphi_j}}{\sqrt{q}} \right) \left( t - \frac{e^{-i\varphi_j}}{\sqrt{q}} \right) = \\ &= \frac{\nu}{\binom{2k}{d}} \prod_{j=1}^q (\sqrt{q}t - e^{i\varphi_j})(\sqrt{q}t - e^{-i\varphi_j}) = \frac{\nu}{\binom{2k}{d}} \prod (qt^2 - 2\sqrt{q}t \cos \varphi_j + 1). \end{aligned}$$

In particular,

$$1 = P_C(1) = \frac{\nu}{\binom{2k}{d}} \prod_{j=1}^q (q - 2\sqrt{q} \cos \varphi_j + 1).$$

Bearing in mind that  $\cos \varphi_j \in [-1, 1]$ , one estimates

$$q - 2\sqrt{q} \cos \varphi_j + 1 \geq (\sqrt{q} - 1)^2$$

and concludes that

$$1 = \frac{\nu}{\binom{2k}{d}} \prod_{j=1}^q (q - 2\sqrt{q} \cos \varphi_j + 1) \geq \frac{\nu}{\binom{2k}{d}} (\sqrt{q} - 1)^{2g}.$$

As a result, there follows

$$q \leq \left( \sqrt[2g]{\frac{\binom{2k}{d}}{\nu}} + 1 \right)^2.$$

By assumption,  $C$  is of minimum distance  $d$ , so that  $\nu \geq 1$  and

$$\left( \sqrt[2g]{\frac{\binom{2k}{d}}{\nu}} + 1 \right)^2 \leq \left( \sqrt[2g]{\binom{2k}{d}} + 1 \right)^2.$$

□

**Proposition 6.** *The following conditions are equivalent for a linear code  $C \subset \mathbb{F}_q^n$ :*

(i)  *$C$  is formally self-dual, i.e., the zeta polynomial  $P_C(t)$  of  $C$  satisfies the functional equation*

$$P_C(t) = P_C\left(\frac{1}{qt}\right) q^g t^{2g}$$

*of the Hasse-Weil polynomial of the function field of a curve of genus  $g$  over  $\mathbb{F}_q$ ;*

(ii) *Duursma's reduced polynomial  $D_C(t) = \sum_{i=0}^{g+g^\perp-2} c_i t^i$  satisfies the functional equation*

$$D_C(t) = D_C\left(\frac{1}{qt}\right) q^{g-1} t^{2g-2} \quad (27)$$

*of the Hasse-Weil polynomial of the function field of a curve of genus  $g-1$  over  $\mathbb{F}_q$ ;*

(iii) *the coefficients of Duursma's reduced polynomial  $D_C(t) = \sum_{i=0}^{g+g^\perp-2} c_i t^i$  of  $C$  satisfy the equalities*

$$c_{g-1+i} = q^i c_{g-1-i} \quad \text{for } \forall 1 \leq i \leq g-1; \quad (28)$$

(iv) *the dual code  $C^\perp \subset \mathbb{F}_q^n$  of  $C$  has dimension  $\dim_{\mathbb{F}_q} C^\perp = \dim_{\mathbb{F}_q} C = k$ , genus  $g(C^\perp) = g(C) = g$  and the homogeneous weight enumerator of  $C$  is*

$$\mathcal{W}_C(x, y) = \mathcal{M}_{2k, k+1}(x, y) + \sum_{j=0}^{g-1} c_{g-1-j} w_j(x, y), \quad (29)$$

where

$$w_j(x, y) := (q-1) \binom{2k}{k+j} \left[ (x-y)^{k+j} y^{k-j} + q^j (x-y)^{k-j} y^{k+j} \right] \quad (30)$$

for  $1 \leq j \leq g-1$ .

$$w_0(x, y) := (q-1) \binom{2k}{k} (x-y)^k y^k. \quad (31)$$

(v) *the dual code  $C^\perp \subset \mathbb{F}_q^n$  of  $C$  has dimension  $\dim_{\mathbb{F}_q} C^\perp = \dim_{\mathbb{F}_q} C = k$ , genus  $g(C^\perp) = g(C) = g$  and the homogeneous weight enumerator*

$$\mathcal{W}_C(x, y) = \mathcal{M}_{2k, k+1}(x, y) + \sum_{w=d}^{k-1} \mathcal{W}_C^{(w)} \varphi_w(x, y) + \mathcal{W}_C^{(k)} (x-y)^k y^k \quad (32)$$



with

$$\varphi_w(x, y) := \sum_{s=w}^{k-1} \binom{2k-w}{s-w} \left[ (x-y)^{2k-s} y^s + q^{k-s} (x-y)^s y^{2k-s} \right] + \binom{2k-w}{k} (x-y)^k y^k \quad (33)$$

for  $d \leq w \leq k-1$ , so that  $C$  can be obtained from an MDS-code of the same length  $2k$  and dimension  $k$  by removing and adjoining appropriate words, depending explicitly on the numbers  $\mathcal{W}_C^{(d)}, \mathcal{W}_C^{(d+1)}, \dots, \mathcal{W}_C^{(k)}$  of the codeword of  $C$  of weight  $\leq k = \dim_{\mathbb{F}_q} C$ .

*Proof.* Towards  $(i) \Rightarrow (ii)$ , one substitutes by  $P_C(t) = (1-t)(1-qt)D_C(t) + t^g$  in (21), in order to obtain

$$(1-t)(1-qt)D_C(t) + t^g = (qt-1)(t-1) \left[ D_C \left( \frac{1}{qt} \right) q^{g-1} t^{2g-2} \right] + t^g,$$

whereas (27).

Conversely,  $(ii) \Rightarrow (i)$  is justified by

$$\begin{aligned} P_C(t) &= (1-t)(1-qt)D_C(t) + t^g = \\ &= (t-1)(qt-1) \left[ D_C \left( \frac{1}{qt} \right) q^{g-1} t^{2g-2} \right] + t^g \\ &= \left[ \left( 1 - \frac{1}{t} \right) t \right] \left[ \left( 1 - \frac{1}{qt} \right) qt \right] \left[ D_C \left( \frac{1}{qt} \right) q^{g-1} t^{2g-2} \right] + \frac{q^g t^{2g}}{q^g t^g} \\ &= \left[ \left( 1 - \frac{q}{qt} \right) \left( 1 - \frac{1}{qt} \right) D_C \left( \frac{1}{qt} \right) + \frac{1}{(qt)^g} \right] q^g t^{2g} = P_C \left( \frac{1}{qt} \right) q^g t^{2g}. \end{aligned}$$

That proves the equivalence  $(i) \Leftrightarrow (ii)$ .

Towards  $(ii) \Leftrightarrow (iii)$ , note that the functional equation of  $D_C(t)$  reads as

$$\begin{aligned} \sum_{i=0}^{2g-2} c_i t^i &= D_C(t) = D_C \left( \frac{1}{qt} \right) q^{g-1} t^{2g-2} = \left( \sum_{i=0}^{2g-2} \frac{c_i}{q^i t^i} \right) q^{g-1} t^{2g-2} \\ &= \sum_{i=0}^{2g-2} c_i q^{g-1-i} t^{2g-2-i} = \sum_{j=0}^{2g-2} c_{2g-2-j} q^{-g+1+j} t^j. \end{aligned}$$

Comparing the coefficients of the left-most and the right-most side, one expresses the formal self-duality of  $C$  by the relations

$$c_j = q^{-g+1+j} c_{2g-2-j} \quad \text{for } \forall 0 \leq j \leq 2g-2.$$

Let  $i := g-1-j$ , in order to express the above conditions in the form

$$c_{g-1+i} = q^i c_{g-1-i} \quad \text{for } \forall -g+1 \leq i \leq g-1. \quad (34)$$

For any  $-g+1 \leq i \leq -1$  note that  $c_{g-1+i} = q^i c_{g-1-i}$  is equivalent to  $c_{g-1-i} = q^{-i} c_{g-1+i}$  and follows from (34) with  $1 \leq -i \leq g-1$ . In the case of  $i = 0$ , (34) holds trivially and (34) amounts to (28). That proves the equivalence of  $(ii)$  with  $(iii)$ .

Towards (iii)  $\Rightarrow$  (iv), one introduces a new variable  $z := x - y$  and expresses (8) in the form

$$\begin{aligned}\mathcal{V}_C(y+z, y) &:= \mathcal{W}_C(y+z, y) - \mathcal{M}_{2k, k+1}(y+z, y) = (q-1) \sum_{i=0}^{2g-2} c_i \binom{2k}{d+i} y^{d+i} z^{2k-d-i} \\ &= (q-1) \sum_{i=0}^{g-1} c_i \binom{2k}{d+i} y^{d+i} z^{2k-d-i} + (q-1) \sum_{i=g}^{2g-2} c_i \binom{2k}{d+i} y^{d+i} z^{2k-d-i}.\end{aligned}$$

Let us change the summation index of the first sum to  $0 \leq j := g-1-i \leq g-1$ , put  $1 \leq j := i-g+1 \leq g-1$  in the second sum and make use of  $d+g = k+1$ , in order to obtain

$$\begin{aligned}\mathcal{V}_C(y+z, y) &= (q-1) \sum_{j=0}^{g-1} c_{g-1-j} \binom{2k}{k-j} y^{k-j} z^{k+j} + (q-1) \sum_{j=1}^{g-1} c_{j+g-1} \binom{2k}{k+j} y^{k+j} z^{k-j}.\end{aligned}\quad (35)$$

Extracting the term with  $j = 0$  from the first sum, one expresses

$$\begin{aligned}\mathcal{V}_C(y+z, y) &= (q-1) c_{g-1} \binom{2k}{k} y^k z^k \\ &\quad + \sum_{j=1}^{g-1} (q-1) \binom{2k}{k+j} \left[ c_{g-1-j} y^{k-j} z^{k+j} + c_{g-1+j} y^{k+j} z^{k-j} \right]\end{aligned}\quad (36)$$

for an arbitrary  $\mathbb{F}_q$ -linear code  $C \subset \mathbb{F}_q^n$ . If  $C$  is formally self-dual, then plugging in by (28) in (36) and making use of (30), (31), one gets

$$\mathcal{V}_C(y+z, y) = \sum_{j=0}^{g-1} c_{g-1-j} w_j(y+z, y).$$

Substituting  $z := x - y$  and  $\mathcal{V}_C(x, y) := \mathcal{W}_C(x, y) - \mathcal{M}_{2k, k+1}(x, y)$ , one derives the equality (29) for the homogeneous weight enumerator of a formally self-dual linear code  $C \subset \mathbb{F}_q^{2k}$ .

In order to justify that (iv) suffices for the formal self-duality of  $C$ , we use that (29) with (30) and (31) is equivalent to

$$\begin{aligned}\mathcal{V}_C(y+z, y) &= \sum_{j=1}^{g-1} c_{g-1-j} (q-1) \binom{2k}{k+j} y^{k-j} z^{k+j} \\ &\quad + c_{g-1} (q-1) \binom{2k}{k} y^k z^k + \sum_{j=1}^{g-1} c_{g-1+j} (q-1) \binom{2k}{k+j} y^{k+j} z^{k-j}\end{aligned}\quad (37)$$

Comparing the coefficients of  $y^{k+j} z^{k-j}$  with  $1 \leq j \leq g-1$  from (36) and (37), one concludes that

$$c_{g-1+j} = c_{g-1-j} q^j \quad \text{for } \forall 1 \leq j \leq g-1.$$

These are exactly the relations (28) and imply the formal self-duality of  $C$ .

Towards  $(iv) \Leftrightarrow (v)$ , it suffices to put  $\mathcal{E}(x, y) := \sum_{j=0}^{g-1} c_{g-1-j} w_j(x, y)$  and to derive that  $\mathcal{E}(x, y) = \sum_{w=d}^{k-1} \mathcal{W}_C^{(w)} \varphi_w(x, y) + \mathcal{W}_C^{(k)} (x-y)^k y^k$ . More precisely, introducing  $i := g-1-j$ , one expresses

$$\begin{aligned} \mathcal{E}(x, y) &= \sum_{i=0}^{g-2} c_i (q-1) \binom{2k}{d+i} \left[ (x-y)^{2k-d-i} y^{d+i} + q^{g-1-i} (x-y)^{d+i} y^{2k-d-i} \right] \\ &\quad + c_{g-1} (q-1) \binom{2k}{k} (x-y)^k y^k. \end{aligned}$$

Plugging in by (11) and exchanging the summation order, one gets

$$\begin{aligned} \mathcal{E}(x, y) &= \sum_{w=d}^{k-1} \sum_{i=w-d}^{g-2} \binom{2k-w}{d+i-w} \mathcal{W}_C^{(w)} [(x-y)^{2k-d-i} y^{d+i} + q^{g-1-i} (x-y)^{d+i} y^{2k-d-i}] \\ &\quad + \sum_{w=d}^k \binom{2k-w}{k} \mathcal{W}_C^{(w)} (x-y)^k y^k. \end{aligned}$$

Introducing  $s := d+i$  and extracting  $\mathcal{W}_C^{(w)}$  as coefficients, one obtains

$$\mathcal{E}(x, y) = \sum_{w=d}^{k-1} \mathcal{W}_C^{(w)} \varphi_w(x, y) + \mathcal{W}_C^{(k)} (x-y)^k y^k.$$

□

Let  $C \subset \mathbb{F}_q^n$  be an  $\mathbb{F}_q$ -linear code of genus  $g$ , whose dual  $C^\perp \subset \mathbb{F}_q^n$  is of genus  $g^\perp$ . In [1], Dodunekov and Landgev introduce the near-MDS linear codes  $C$  as the ones with zeta polynomial  $P_C(t) \in \mathbb{Q}[t]$  of degree  $\deg P_C(t) := g + g^\perp = 2$ . Thus,  $C$  is a near-MDS code if and only if it has constant Duursma's reduced polynomial  $D_C(t) = c_0 \in \mathbb{Q}$ . Kim and Hyun prove in [7]) that a near-MDS code  $C$  satisfies the Riemann Hypothesis Analogue exactly when

$$\frac{1}{(\sqrt{q}+1)^2} \leq c_0 \leq \frac{1}{(\sqrt{q}-1)^2}.$$

The next proposition characterizes the formally-self-dual codes  $C \subset \mathbb{F}_q^n$  of genus 2, which satisfy the Riemann Hypothesis Analogue. By Proposition 6 (ii),  $C$  is a formally self-dual linear code of genus 2 exactly when its Duursma's reduced polynomial is

$$D_C(t) = c_0 + c_1 t + q c_0 t^2$$

for some  $c_0, c_1 \in \mathbb{Q}$ ,  $0 < c_0 < 1$ .

**Proposition 7.** *A formally self-dual linear code  $C \subset \mathbb{F}_q^{2k}$  with a quadratic Duursma's reduced polynomial  $D_C(t) = c_0 + c_1 t + q c_0 t^2 \in \mathbb{Q}[t]$ ,  $0 < c_0 < 1$  satisfies the Riemann Hypothesis Analogue if and only if*

$$[(q+1)c_0 + c_1]^2 \geq 4c_0, \tag{38}$$

$$q - 4\sqrt{q} + 1 \leq \frac{c_1}{c_0} \leq q + 4\sqrt{q} + 1, \quad (39)$$

$$c_1 \leq \min \left( \frac{1}{(\sqrt{q} - 1)^2} - 2\sqrt{q}c_0, \frac{1}{(\sqrt{q} + 1)^2} + 2\sqrt{q}c_0 \right). \quad (40)$$

*Proof.* According to (22) from the proof of Proposition 4, the zeta polynomial

$$P_C(t) = (1 - t)(1 - qt)(qc_0t^2 + c_1t + c_0) + t^2$$

satisfies the Riemann Hypothesis Analogue if and only if there exist  $\varphi, \psi \in [0, 2\pi)$  with

$$P_C(t) = q^2c_0 \left( t - \frac{e^{i\varphi}}{\sqrt{q}} \right) \left( t - \frac{e^{-i\varphi}}{\sqrt{q}} \right) \left( t - \frac{e^{i\psi}}{\sqrt{q}} \right) \left( t - \frac{e^{-i\psi}}{\sqrt{q}} \right).$$

Comparing the coefficients of  $t$  and  $t^2$  from  $P_C(t)$ , one expresses this condition by the equalities

$$\begin{aligned} c_1 - (q + 1)c_0 &= -2\sqrt{q}c_0[\cos(\varphi) + \cos(\psi)], \\ 1 + 2qc_0 - (q + 1)c_1 &= 2qc_0[1 + 2\cos(\varphi)\cos(\psi)]. \end{aligned}$$

These are equivalent to

$$\cos(\varphi) + \cos(\psi) = \frac{(q + 1)c_0 - c_1}{2\sqrt{q}c_0}$$

and

$$\cos(\varphi)\cos(\psi) = \frac{1 - (q + 1)c_1}{4qc_0}.$$

In other words, the quadratic equation

$$f(t) := t^2 + \frac{c_1 - (q + 1)c_0}{2\sqrt{q}c_0}t + \frac{1 - (q + 1)c_1}{4qc_0} \in \mathbb{Q}[t]$$

has roots  $-1 \leq t_1 = \cos(\varphi) \leq t_2 = \cos(\psi) \leq 1$ . This, in turn, holds exactly when the discriminant

$$D(f) = \left[ \frac{c_1 - (q + 1)c_0}{2\sqrt{q}c_0} \right]^2 - \frac{4[1 - (q + 1)c_1]}{4qc_0} \geq 0 \quad (41)$$

is non-negative, the vertex

$$-1 \leq \frac{(q + 1)c_0 - c_1}{4\sqrt{q}c_0} \leq 1 \quad (42)$$

belongs to the segment  $[-1, 1]$  and the values of  $f(t)$  at the ends of this segment are non-negative,

$$f(1) \geq 0, \quad f(-1) \geq 0. \quad (43)$$

The equivalence of (41) to (38) is straightforward. Since  $C$  is of minimum distance  $d = k - 1$  and  $\mathcal{W}_C^{(k-1)} = (q - 1)\binom{2k}{k-1}c_0 \in \mathbb{N}$ , the constant term  $c_0 > 0$  of  $D_C(t)$  is a

positive rational number and one can multiply (42) by  $-4\sqrt{q}c_0 < 0$ , add  $(q+1)c_0$  to all the terms and rewrite it in the form

$$(q - 4\sqrt{q} + 1)c_0 \leq c_1 \leq (q + 4\sqrt{q} + 1)c_0.$$

Making use of  $c_0 > 0$ , one observes that the above inequalities are tantamount to (39). Finally,

$$4qc_0f(1) = 4qc_0 + 2\sqrt{q}[c_1 - (q+1)c_0] + 1 - (q+1)c_1 = (-c_1 - 2\sqrt{q}c_0)(\sqrt{q} - 1)^2 + 1 \geq 0$$

and

$$4qc_0f(-1) = 4qc_0 - 2\sqrt{q}[c_1 - (q+1)c_0] + 1 - (q+1)c_1 = (2\sqrt{q}c_0 - c_1)(\sqrt{q} + 1)^2 + 1 \geq 0$$

can be expressed as (40). □

### 3 Duursma's reduced polynomial of a function field

Let  $F = \mathbb{F}_q(X)$  be the function field of a curve  $X$  of genus  $g$  over  $\mathbb{F}_q$  and  $h_g := h(F)$  be the class number of  $F$ , i.e., the number of the linear equivalence classes of the divisors of  $F$  of degree 0. The present section introduces an additive decomposition of the Hasse-Weil polynomial  $L_F(t) \in \mathbb{Z}[t]$  of  $F$ , which associates to  $F$  a sequence  $\{h_i\}_{i=1}^{g-1}$  of virtual class numbers  $h_i$  of function fields of curves of genus  $i$  over  $\mathbb{F}_q$ .

**Lemma 8.** *The following conditions are equivalent for a polynomial  $L_g(t) \in \mathbb{Q}[t]$  of degree  $\deg L_g(t) = 2g$ :*

(i)  $L_g(t)$  satisfies the functional equation

$$L_g(t) = L_g\left(\frac{1}{qt}\right) q^g t^{2g}$$

of the Hasse-Weil polynomial of the function field of a curve of genus  $g$  over  $\mathbb{F}_q$ ;

$$(ii) \quad L_{g-1}(t) := \frac{L_g(t) - L_g(1)t^g}{(1-t)(1-qt)}$$

is a polynomial with rational coefficients of degree  $2g-2$ , satisfying the functional equation

$$L_{g-1}(t) = L_{g-1}\left(\frac{1}{qt}\right) q^{g-1} t^{2g-2}$$

of the Hasse-Weil polynomial of the function field of a curve of genus  $g-1$  over  $\mathbb{F}_q$ ;

$$(iii) \quad L_g(t) = \sum_{i=0}^g h_i t^i (1-t)^{g-i} (1-qt)^{g-i}$$

for some rational numbers  $h_i \in \mathbb{Q}$ .

*Proof.* Towards (i)  $\Rightarrow$  (ii), let us note that the polynomial  $M_g(t) := L_g(t) - L_g(1)t^g$  vanishes at  $t = 1$ , so that it is divisible by  $1 - t$ . Further,

$$M_g(t) = L_g(t) - L_g(1)t^g = \left[ L_g\left(\frac{1}{qt}\right) - \frac{L_g(1)}{q^g t^g} \right] q^g t^{2g} = M_g\left(\frac{1}{qt}\right) q^g t^{2g}$$

satisfies the functional equation of the Hasse-Weil polynomial of the function field of a curve of genus  $g$  over  $\mathbb{F}_q$ . In particular,  $M_g\left(\frac{1}{q}\right) = M_g(1)\frac{q^g}{q^{2g}} = 0$  and  $M_g(t)$  is divisible by the linear polynomial  $q\left(\frac{1}{q} - t\right) = 1 - qt$ , which is relatively prime to  $1 - t$  in  $\mathbb{Q}[t]$ . As a result,

$$L_{g-1}(t) := \frac{M_g(t)}{(1-t)(1-qt)} \in \mathbb{Q}[t]$$

is a polynomial of degree  $\deg L_{g-1}(t) = 2g - 2$ . Straightforwardly,

$$\begin{aligned} L_{g-1}\left(\frac{1}{qt}\right) q^{g-1} t^{2g-2} &= \left[ M_g\left(\frac{1}{qt}\right) : \left(1 - \frac{1}{qt}\right) \left(1 - \frac{1}{t}\right) \right] \\ &= \frac{M_g(t)}{qt^2} : \frac{(qt-1)(t-1)}{qt^2} = \frac{M_g(t)}{(1-t)(1-qt)} = L_{g-1}(t) \end{aligned}$$

satisfies the functional equation of the Hasse-Weil polynomial of the function field of a curve of genus  $g - 1$  over  $\mathbb{F}_q$ .

The implication (ii)  $\Rightarrow$  (i) follows from the functional equation of  $L_{g-1}(t)$ , applied to  $L_g(t) = (1-t)(1-qt)L_{g-1}(t) + L_g(1)t^g$ . Namely,

$$\begin{aligned} &L_g\left(\frac{1}{qt}\right) q^g t^{2g} \\ &= \left[ \left(1 - \frac{1}{qt}\right) qt \right] \left[ \left(1 - \frac{1}{t}\right) t \right] \left[ L_{g-1}\left(\frac{1}{qt}\right) q^{g-1} t^{2g-2} \right] + \frac{L_g(1)}{q^g t^g} q^g t^{2g} \\ &= (qt-1)(t-1)L_{g-1}(t) + L_g(1)t^g \\ &= (1-t)(1-qt)L_{g-1}(t) + L_g(1)t^g = L_g(t). \end{aligned}$$

We derive (i)  $\Rightarrow$  (iii) by an induction on  $g$ , making use of (ii). More precisely, for  $g = 1$  one has  $L_0(t) := \frac{L_1(t) - L_1(1)t}{(1-t)(1-qt)} \in \mathbb{Q}[t]$  of degree  $\deg L_0(t) = 0$  or  $L_0 \in \mathbb{Q}$ . Then

$$L_1(t) = (1-t)(1-qt)L_0 + L_1(1)t = \sum_{i=0}^1 h_i t^i (1-t)^{1-i} (1-qt)^{1-i}$$

with  $h_0 := L_0 \in \mathbb{Q}$  and  $h_1 := L_1(1) \in \mathbb{Q}$ . In the general case, (ii) provides a polynomial

$$L_{g-1}(t) := \frac{L_g(t) - L_g(1)t^g}{(1-t)(1-qt)},$$

subject to the functional equation

$$L_{g-1}(t) = L_{g-1}\left(\frac{1}{qt}\right) q^{g-1} t^{2g-2}$$

of the Hasse-Weil polynomial of the function field of a curve of genus  $g - 1$  over  $\mathbb{F}_q$ . By the inductual hypothesis, there exist  $h'_i \in \mathbb{Q}$ ,  $0 \leq i \leq g - 1$  with

$$L_{g-1}(t) = \sum_{i=0}^{g-1} h'_i t^i (1-t)^{g-1-i} (1-qt)^{g-1-i}.$$

Then

$$L_g(t) = (1-t)(1-qt)L_{g-1}(t) + L_g(1)t^g = \sum_{i=0}^g h_i t^i (1-t)^{g-i} (1-qt)^{g-i}$$

with  $h_i := h'_i \in \mathbb{Q}$  for  $0 \leq i \leq g - 1$  and  $h_g := L_g(1) \in \mathbb{Q}$  justifies (i)  $\Rightarrow$  (iii).

Towards (iii)  $\Rightarrow$  (i), let us assume that  $L_g(t) = \sum_{i=0}^g h_i t^i (1-t)^{g-i} (1-qt)^{g-i}$ . Then

$$\begin{aligned} L\left(\frac{1}{qt}\right) q^g t^{2g} &= \left[ \sum_{i=0}^g \frac{h_i}{q^i t^i} \left(1 - \frac{1}{qt}\right)^{g-i} \left(1 - \frac{1}{t}\right)^{g-i} \right] q^g t^{2g} \\ &= \sum_{i=0}^g \left[ \frac{h_i}{q^i t^i} q^i t^{2i} \right] \left[ \left(1 - \frac{1}{qt}\right) qt \right]^{g-i} \left[ \left(1 - \frac{1}{t}\right) t \right]^{g-i} \\ &= \sum_{i=0}^g h_i t^i (qt - 1)^{g-i} (t - 1)^{g-i} = L_g(t) \end{aligned}$$

satisfies the functional equation of the Hasse-Weil polynomial of the function field of a curve of genus  $g$  over  $\mathbb{F}_q$ . □

**Proposition 9.** *Let  $F = \mathbb{F}_q(X)$  be the function field of a smooth irreducible curve  $X/\mathbb{F}_q \subset \mathbb{P}^N(\overline{\mathbb{F}_q})$  of genus  $g$ , defined over  $\mathbb{F}_q$ , with  $h(F)$  linear equivalence classes of divisors of degree 0,  $\mathcal{A}_i$  effective divisors of degree  $i \geq 0$ , Hasse-Weil polynomial  $L_F(t) \in \mathbb{Q}[t]$  and Duursma's reduced polynomial  $D_F(t) \in \mathbb{Q}[t]$ , defined by the equality*

$$L_F(t) = (1-t)(1-qt)D_F(t) + h(F)t^g.$$

Then:

- (i)  $D_F(t) = \sum_{i=0}^{g-2} \mathcal{A}_i (t^i + q^{g-1-i} t^{2g-2-i}) + \mathcal{A}_{g-1} t^{g-1} \in \mathbb{Z}[t]$  is a polynomial with integral coefficients, which is uniquely determined by  $\mathcal{A}_0 = 1, \mathcal{A}_1, \dots, \mathcal{A}_{g-1}$ ;
- (ii) the equality

$$\frac{D_F(t)}{(1-t)(1-qt)} = \sum_{i=0}^{\infty} \mathcal{B}_i t^i \tag{44}$$

of formal power series of  $t$  holds for

$$\mathcal{B}_i = \sum_{j=0}^i \mathcal{A}_j \left( \frac{q^{i-j+1} - 1}{q - 1} \right) \tag{45}$$

for  $0 \leq i \leq g-1$ ,

$$\mathcal{B}_i = \sum_{j=0}^{g-1} \mathcal{A}_j \left( \frac{q^{i-j+1} - 1}{q-1} \right) + \sum_{j=g}^i \mathcal{A}_{2g-2-j} \left( \frac{q^{i-g+2} - q^{j-g+1}}{q-1} \right) \quad (46)$$

for  $g \leq i \leq 2g-3$ ,

$$\mathcal{B}_i = D_F(1) \left( \frac{q^{i-g+2} - 1}{q-1} \right) \quad (47)$$

for  $i \geq 2g-2$ ;

(iii) the natural numbers  $\mathcal{B}_i$ ,  $i \geq 0$  from (ii) satisfy the relations

$$\mathcal{B}_i = q^{i-g+2} \mathcal{B}_{2g-4-i} + D_F(1) \left( \frac{q^{i-g+2} - 1}{q-1} \right) \quad \text{for } \forall g-1 \leq i \leq 2g-4; \quad (48)$$

$$\mathcal{B}_i = D_F(1) \left( \frac{q^{i-g+2} - 1}{q-1} \right) \quad \text{for } \forall i \geq 2g-3. \quad (49)$$

(iv) the number  $h(F)$  of the linear equivalence classes of the divisors of  $F$  of degree 0 satisfies the inequalities

$$(\sqrt{q} - 1)^{2g} \leq h(F) \leq (\sqrt{q} + 1)^{2g}$$

*Proof.* (i) By Theorem 4.1.6. (ii) and Theorem 4.1.11 from [8], the Hasse-Weil zeta function of  $F$  is the generating function

$$Z_F(t) = \frac{L_F(t)}{(1-t)(1-qt)} = \sum_{j=0}^{\infty} \mathcal{A}_j t^j$$

of the sequence  $\{\mathcal{A}_i\}_{i=0}^{\infty}$ . According to Lemma 8 and  $L_F(1) = h(F)$ ,

$$D_F(t) := \frac{L_F(t) - h(F)t^g}{(1-t)(1-qt)}$$

is a polynomial of  $\deg D_F(t) = 2g-2$ , subject to the functional equation of the Hasse-Weil polynomial of the function field of a curve of genus  $g-1$  over  $\mathbb{F}_q$ . Thus,

$$Z_F(t) = D_F(t) + \frac{h(F)t^g}{(1-t)(1-qt)} = \sum_{j=0}^{\infty} \mathcal{A}_j t^j. \quad (50)$$

Let  $l(G)$  is the dimension of the space  $H^0(X, \mathcal{O}_X(G))$  of the global holomorphic sections of the line bundle  $\mathcal{O}_X(G) \rightarrow X$ , associated with a divisor  $G \in \text{Div}(F)$ . Riemann-Roch Theorem asserts that

$$l(G) = l(K_X - G) + \deg(G) - g + 1$$

for a canonical divisor  $K_X$  of  $X$ . For any  $j \geq g-1$ , suppose that  $G_1, \dots, G_{h(F)} \in \text{Div}(F)$  is a complete set of representatives of the linear equivalence classes of the divisors of  $F$  of degree  $j$ . Then

$$\mathcal{A}_j = \sum_{\nu=1}^{h(F)} \frac{q^{l(G_\nu)} - 1}{q-1} = q^{j-g+1} \sum_{\nu=1}^{h(F)} \left( \frac{q^{l(K_X - G_\nu)} - 1}{q-1} \right) + h(F) \left( \frac{q^{j-g+1} - 1}{q-1} \right) \quad (51)$$



for  $g \leq j \leq 2g - 2$  and

$$\mathcal{A}_j = h(F) \left( \frac{q^{j-g+1} - 1}{q - 1} \right) \quad \text{for } \forall j \geq 2g - 1. \quad (52)$$

Note that  $K_Y - G_1, \dots, K_Y - G_{h(F)}$  is a complete set of representatives of the linear equivalence classes of the divisors of  $F$  of degree  $2g - 2 - j$ , so that

$$\mathcal{A}_{2g-2-j} = \sum_{\nu=1}^{h(F)} \frac{q^{l(K_Y - G_\nu)} - 1}{q - 1}. \quad (53)$$

Plugging in by (53) in (51), one obtains

$$\mathcal{A}_j = q^{j-g+1} \mathcal{A}_{2g-2-j} + h(F) \left( \frac{q^{j-g+1} - 1}{q - 1} \right) \quad \text{for } g \leq j \leq 2g - 2, \quad (54)$$

whereas

$$Z_F(t) = \sum_{j=0}^{g-1} \mathcal{A}_j t^j + \sum_{j=g}^{2g-2} q^{j-g+1} \mathcal{A}_{2g-2-j} t^j + h(F) \sum_{j=g}^{\infty} \left( \frac{q^{j-g+1} - 1}{q - 1} \right) t^j,$$

Putting  $i := 2g - 2 - j$  in the second sum and  $i := j - g$  in the third sum, one expresses

$$\begin{aligned} Z_F(t) &= \sum_{i=0}^{g-2} \mathcal{A}_i (t^i + q^{g-1-i} t^{2g-2-i}) + \mathcal{A}_{g-1} t^{g-1} \\ &\quad + h(F) \left[ \frac{qt^g}{q-1} \left( \sum_{i=0}^{\infty} q^i t^i \right) - \frac{t^g}{q-1} \left( \sum_{i=0}^{\infty} t^i \right) \right], \end{aligned}$$

Summing up the geometric progressions

$$\sum_{i=0}^{\infty} q^i t^i = \frac{1}{1-qt}, \quad \sum_{i=0}^{\infty} t^i = \frac{1}{1-t},$$

one derives

$$Z_F(t) = \sum_{i=0}^{g-2} \mathcal{A}_i (t^i + q^{g-1-i} t^{2g-2-i}) + \mathcal{A}_{g-1} t^{g-1} + h(F) \frac{t^g}{(1-t)(1-qt)},$$

whereas

$$D_F(t) = \sum_{i=0}^{g-2} \mathcal{A}_i (t^i + q^{g-1-i} t^{2g-2-i}) + \mathcal{A}_{g-1} t^{g-1}.$$

In particular,  $D_F(t) \in \mathbb{Z}[t]$  has integral coefficients.

(ii) Let us expand

$$\frac{1}{1-t} = \sum_{i=0}^{\infty} t^i, \quad \frac{1}{1-qt} = \sum_{i=0}^{\infty} q^i t^i$$

as sums of geometric progressions and note that

$$\frac{1}{(1-t)(1-qt)} = \sum_{i=0}^{\infty} (1+q+\dots+q^i)t^i = \sum_{i=0}^{\infty} \left( \frac{q^{i+1}-1}{q-1} \right) t^i.$$

Then represent Duursma's reduced polynomial in the form

$$D_F(t) = \sum_{j=0}^{g-1} \mathcal{A}_j t^j + \sum_{j=g}^{2g-2} \mathcal{A}_{2g-2-j} q^{j-g+1} t^j. \quad (55)$$

Now, the comparison of the coefficients of  $t^i$ ,  $i \geq 0$  from the left hand side and the right hand side of (44) provides (45), (46) and

$$\mathcal{B}_i = \sum_{j=0}^{g-1} \mathcal{A}_j \left( \frac{q^{i-j+1}-1}{q-1} \right) + \sum_{j=g}^{2g-2} \mathcal{A}_{2g-2-j} q^{j-g+1} \left( \frac{q^{i-j+1}-1}{q-1} \right) \quad \text{for } i \geq 2g-2.$$

The last formula can be expressed in the form

$$\begin{aligned} \mathcal{B}_i &= \\ &= \frac{q^{i+1}}{q-1} \left( \sum_{j=0}^{g-1} \mathcal{A}_j q^{-j} + \sum_{j=g}^{2g-2} \mathcal{A}_{2g-2-j} q^{j-g+1} q^{-j} \right) - \frac{1}{q-1} \left( \sum_{j=0}^{g-1} \mathcal{A}_j + \sum_{j=g}^{2g-2} \mathcal{A}_{2g-2-j} q^{j-g+1} \right) \\ &= \frac{q^{i+1}}{q-1} D_F \left( \frac{1}{q} \right) - \frac{1}{q-1} D_F(1). \end{aligned}$$

According to Lemma 8 (i)  $\Rightarrow$  (ii), Duursma's reduced polynomial of  $F$  satisfies the functional equation  $D_F(t) = D_F \left( \frac{1}{qt} \right) q^{g-1} t^{2g-2}$ . In particular,  $D_F(1) = D_F \left( \frac{1}{q} \right) q^{g-1}$  and there follows (47).

(iii) Due to  $\mathcal{A}_i \geq 0$  for  $\forall i \geq 0$ ,  $\mathcal{B}_i$  are sums of non-negative integers. Moreover,  $\mathcal{B}_i \geq \mathcal{A}_i \left( \frac{q^{i+1}}{q-1} \right) \geq \mathcal{A}_0 = 1 > 0$  for  $\forall i \geq 0$  reveals that all  $\mathcal{B}_i$  are natural numbers.

Towards (48), let us introduce the polynomial  $\psi(t) := \sum_{j=0}^{g-2} \mathcal{A}_j t^j \in \mathbb{Z}[t]$  and express

$$\begin{aligned} D_F(t) &= \sum_{j=0}^{g-2} \mathcal{A}_j t^j + q^{g-1} t^{2g-2} \left[ \sum_{j=0}^{g-2} \mathcal{A}_j (qt)^{-j} \right] + \mathcal{A}_{g-1} t^{g-1} \\ &= \psi(t) + \psi \left( \frac{1}{qt} \right) q^{g-1} t^{2g-2} + \mathcal{A}_{g-1} t^{g-1}. \end{aligned}$$

In particular,

$$D_F(1) = \psi(1) + \psi \left( \frac{1}{q} \right) q^{g-1} + \mathcal{A}_{g-1}. \quad (56)$$

Straightforwardly,

$$\begin{aligned}
& \mathcal{B}_{g-1} - q\mathcal{B}_{g-3} \\
&= \frac{q^g}{q-1} \left( \sum_{j=0}^{g-2} \mathcal{A}_j q^{-j} \right) - \frac{1}{q-1} \left( \sum_{j=0}^{g-2} \mathcal{A}_j \right) + \mathcal{A}_{g-1} - \\
& \quad - \frac{q^{g-1}}{q-1} \left( \sum_{j=0}^{g-2} \mathcal{A}_j q^{-j} \right) + \frac{q}{q-1} \left( \sum_{j=0}^{g-2} \mathcal{A}_j \right) \\
&= \psi \left( \frac{1}{q} \right) q^{g-1} + \psi(1) + \mathcal{A}_{g-1} = D_F(1).
\end{aligned}$$

That proves (48) for  $i = g-1$ . In the case of  $g \leq i \leq 2g-4$  note that  $0 \leq 2g-4-i \leq g-4$  and

$$\begin{aligned}
& (q-1)(\mathcal{B}_i - q^{i-g+2}\mathcal{B}_{2g-4-i}) \\
&= \sum_{j=0}^{g-1} \mathcal{A}_j (q^{i-j+1} - 1) + \sum_{j=g}^i \mathcal{A}_{2g-2-j} (q^{i-g+2} - q^{j-g+1}) - \sum_{j=0}^{2g-4-i} \mathcal{A}_j (q^{g-1-j} - q^{i-g+2}).
\end{aligned}$$

Changing the summation index of the second sum to  $s := 2g-2-j$ , one obtains

$$\begin{aligned}
& (q-1)(\mathcal{B}_i - q^{i-g+2}\mathcal{B}_{2g-4-i}) \\
&= q^{i+1} \left( \sum_{j=0}^{g-1} \mathcal{A}_j q^{-j} \right) - \left( \sum_{j=0}^{g-1} \mathcal{A}_j \right) + q^{i-g+2} \left( \sum_{s=2g-2-i}^{g-2} \mathcal{A}_s \right) \\
& \quad - q^{g-1} \left( \sum_{s=2g-2-i}^{g-2} \mathcal{A}_s q^{-s} \right) - q^{g-1} \left( \sum_{j=0}^{2g-4-i} \mathcal{A}_j q^{-j} \right) + q^{i-g+2} \left( \sum_{j=0}^{2g-4-i} \mathcal{A}_j \right).
\end{aligned}$$

An appropriate grouping of the sums yields

$$\begin{aligned}
& (q-1)(\mathcal{B}_i - q^{i-g+2}\mathcal{B}_{2g-4-i}) \\
&= \psi \left( \frac{1}{q} \right) q^{i+1} + \mathcal{A}_{g-1} q^{i-g+2} - \psi(1) - \mathcal{A}_{g-1} + \psi(1) q^{i-g+2} - \psi \left( \frac{1}{q} \right) q^{g-1} \\
&= (q^{i-g+2} - 1) \left[ \psi(1) + \psi \left( \frac{1}{q} \right) q^{g-1} + \mathcal{A}_{g-1} \right] = D_F(1)(q^{i-g+2} - 1).
\end{aligned}$$

That justifies (48).

Note that (49) with  $i \geq 2g-2$  coincides with (47). In the case of  $i = 2g-3$ ,

$$(q-1)\mathcal{B}_{2g-3} = \sum_{j=0}^{g-1} \mathcal{A}_j (q^{2g-2-j} - 1) + \sum_{s=1}^{g-2} \mathcal{A}_s (q^{g-1} - q^{g-1-s}),$$

after changing the summation index of the second sum to  $s := 2g - 2 - j$ . Then

$$\begin{aligned}
& (q-1)\mathcal{B}_{2g-3} \\
&= q^{2g-2} \left( \sum_{j=0}^{g-2} \mathcal{A}_j q^{-j} \right) - \left( \sum_{j=0}^{g-2} \mathcal{A}_j \right) + \mathcal{A}_{g-1}(q^{g-1} - 1) + \\
& \quad + q^{g-1} \left( \sum_{j=0}^{g-2} \mathcal{A}_j \right) - q^{g-1} \left( \sum_{j=0}^{g-2} \mathcal{A}_j q^{-j} \right) \\
&= (q^{g-1} - 1) \left[ \psi(1) + \psi \left( \frac{1}{q} \right) q^{g-1} + \mathcal{A}_{g-1} \right] = D_F(1)(q^{g-1} - 1),
\end{aligned}$$

which is tantamount to (49) with  $i = 2g - 3$ .

(iv) By the Hasse-Weil Theorem, all the roots of  $L_F(t)$  belong to the circle  $S \left( \frac{1}{\sqrt{q}} \right) = \left\{ z \in \mathbb{C} \mid |z| = \frac{1}{\sqrt{q}} \right\}$ . The proof of Proposition 4 specifies that

$$L_F(t) = a_{2g} \prod_{j=1}^g \left( t - \frac{e^{i\varphi_j}}{\sqrt{q}} \right) \left( t - \frac{e^{-i\varphi_j}}{\sqrt{q}} \right)$$

for some  $\varphi_j \in [0, 2\pi)$ . The functional equation  $L_F(t) = L_F \left( \frac{1}{qt} \right) q^g t^{2g}$  implies that  $a_{2g} = q^g a_0$ . Combining with  $a_0 = L_F(0) = 1$ , one gets

$$L_F(t) = \prod_{j=1}^g (\sqrt{q}t - e^{i\varphi_j})(\sqrt{q}t - e^{-i\varphi_j}) = \prod_{j=1}^g (qt^2 - 2\sqrt{q} \cos \varphi_j t + 1).$$

The substitution  $t = 1$  provides

$$h(F) = L_F(1) = \prod_{j=1}^g (q - 2\sqrt{q} \cos \varphi_j + 1).$$

However,  $\cos \varphi_j \in [-1, 1]$  requires

$$(\sqrt{q} - 1)^2 \leq q - 2\sqrt{q} \cos \varphi_j + 1 \leq (\sqrt{q} + 1)^2,$$

whereas

$$(\sqrt{q} - 1)^{2g} \leq h(F) = L_F(1) = \prod_{j=1}^g (q - 2\sqrt{q} \cos \varphi_j + 1) \leq (\sqrt{q} + 1)^{2g}.$$

□

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